SOME SURFACES WITH CANONICAL MAP OF DEGREE 4.

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ABSTRACT. In this short note we construct unbounded families of minimal surfaces of general type with canonical map of degree 4 such that the limits of the slopes K^2/χ assume countably many different values in the range $[6.\bar{6}, 8]$.

INTRODUCTION

It is well known since the pioneering work of Beauville [Bea79] and a Theorem of Xiao Gang [Xia86] that the degree of the canonical map of a surface (=complex manifold of dimension 2), if we assume the Euler characteristic $\chi(\mathcal{O})$ big enough, is bounded from above by 8. We address the reader to the beautiful survey of M. Mendes Lopes and R. Pardini [MLP] on the subject.

We read from there, among other things, that there are examples of sequences of surfaces with $\chi(\mathcal{O})$ arbitrarily large (that we call for short from now on *unbounded families*) with canonical map of degree $2, 4, 6, 8$. It seems that we do not know much on what are the possible accumulation points of slopes $K^2/\chi(\mathcal{O})$ for unbounded families of minimal surfaces with canonical map of fixed degree, compare [MLP, Question 5.6].

In particular the only unbounded families of minimal surfaces with canonical map of degree 4 known to us are the product of two hyperelliptic curves (mentioned also in [MLP], they have $K^2/\chi = 8$) and those produced by Gallego and Purnaprajna for which $\lim_{n\to\infty} K^2_{S_n}/\chi(\mathcal{O}(S_n))$ is either 8 or 4, see the last column of [GP08, Table at page 5491]

Inspired by certain constructions of $K3$ surfaces in [GP15], we manage to construct several unbounded families with canonical map of degree 4. We can prove

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This paper originated by several enlighting discussions of the second author with C. Gleißner and C. Rito on the idea of considering product-quotient surfaces $C_1 \times C_2/G$ with a subgroup $H \subset G$ such that the canonical map of $C_1 \times C_2/H$ factors through $C_1 \times C_2/G$. He thanks both of them heartily. We thank J. Stevens, S. Troncoso and the anonymous user "mathlove" on Stack Exchange for interesting discussions on the behaviour of the function σ. The second author is indebted with M. Mendes Lopes and R. Pardini for sharing a preliminary version of their beautiful survey. The second author thanks M. Penegini and F. Polizzi for inviting him to give a talk at the 8th ECM in Portorose, where he presented a preliminary version of this result: the beautiful atmosphere and the clever questions posed there helped us in improving our results.

Theorem. *There are countably many unbounded sequences* $\{S_n\}$ *of surfaces of general type with canonical map of degree* 4 *such that* $\lim_{n\to\infty} K^2_{S_n}/\chi(\mathcal{O}(S_n))$ *assume pairwise distinct values in the range* $[6.\bar{6}, 8]$.

It is not clear to us at the moment how big is the set of pairwise distinct values we can obtain in $[6.\bar{6}, 8]$. We know that it contains all numbers of the form $8 - \frac{1}{m}$, where $m \geq 6$ is not a prime.

This is related to the following question of indipendent interest, whose answer may be known to experts.

Consider a rational number $0 < \frac{k}{n} < 1$. Here we assume $\gcd(k, n) = 1$. Consider the continued fraction

$$\frac{n}{k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}} \tag{0.1}$$

and define

$$\sigma\left(\frac{k}{n}\right) := \frac{1 + \sum(b_j - 1)}{n}.$$

Obviously $\sigma > 0$, $\sigma\left(\frac{1}{n}\right) = 1$. It is known [TU, Lemma 3.3] that $\sigma \leq 1$.

Question: *What is the subset* $\left\{\sigma\left(\frac{k}{n}\right)\right\} \subset [0, 1]$*?*

If this subset would be dense then the subset of $[6.\bar{6}, 8]$ in the Theorem would be dense as well. If this subset would be dense in any interval, then the subset in the Theorem would be dense in some interval as well. Unfortunately this does not seems to be the case.

More precisely we want to know

Question: *What are the possible limits of* $\left\{\sigma\left(\frac{k}{n}\right)\right\} \subset [0, 1]$ *for sequences of rational numbers* $\frac{k}{n}$ *with unbounded denominators?*

It is not difficult to prove that under mild assumptions such sequences converge to zero. Still, there are exceptions: $\lim_{n\to\infty} \sigma\left(\frac{m}{mn+1}\right) = \frac{1}{m}$. We could not obtain any sequence with limit neither zero nor of the form $\frac{1}{m}$.

All these surfaces are product-quotient surfaces. The name product-quotient surfaces has been introduced by the second author and I. Bauer in [BP12] following an idea of Fabrizio Catanese (compare [Cat00], [BC04], [BCGP12]). Their canonical map was studied, in the special case of the surfaces isogenous to a product, in [Cat18]. To our knowledge they were used first for constructing surfaces with canonical map of high degree in [GPR]. We plan to use them to construct more examples in the future.

Notation. For each real number z, let $\lceil z \rceil$ be the smallest integer greater or equal than z.

For each pair of integers $z, n \in \mathbb{N}$ we denote by $[z]_n$ the unique integer, $0 \leq [z]_n \leq n - 1$, such that $z - [z]_n$ is divisible by n.

We say that a point of a complex analytic variety is a *singular point of type* $\frac{p}{q} \in \mathbb{Q}$, with $p \in \mathbb{Z} \setminus \{0\}$, $q \in \mathbb{N} \setminus \{0\}$, $\gcd(p,q) = 1$, if a neighbourhood of it is analytically isomorphic to the quotient of a neighbourhood of the origin of $\mathbb{C}^2$ by the cyclic group generated by the automorphism $(x,y) \mapsto (e^{\frac{2\pi i}{q}} x, e^{p\frac{2\pi i}{q}} y)$. We say that an analytic variety has *basket* of singularities $a_1\frac{p_1}{q_1} + a_2\frac{p_2}{q_2} + \cdots + a_r\frac{p_r}{q_r}$ if its singular locus is finite and can be partitioned in r subsets $S_1, \ldots, S_r$ of respective cardinality $a_1, \ldots, a_r$ such that each point in S_j is a singularity of type $\frac{p_j}{q_j}$.

1. GENERALIZED WIMAN CURVES

Definition 1.1 (Generalized Wiman curves)**.** Let $f \in \mathbb{C}[x_0, x_1]$ be a homogeneous polynomial such that, for all $n \in \mathbb{N}$, $f(x_0^n, x_1^n)$ has no multiple roots. In other words, we are requiring that f has no multiple roots and that neither x_0 nor x_1 divide f. Set d for the degree of f.

Then we consider, for each positive integer $n \geq 1$, the hyperelliptic curve

$$C_{n,d}\colon \left\{ y^2 = x_0^{[nd]_2} f(x_0^n, x_1^n) \right\} \subset \mathbb{P}\left(1, 1, \left\lceil \frac{nd}{2} \right\rceil \right)$$

The genus of $C_{n,d}$ is $\lceil \frac{nd}{2} \rceil - 1$. We consider the following automorphisms of $C_{n,d}$:

$$\iota = \iota_{n,d}\colon (x_0, x_1, y) \mapsto (x_0, x_1, -y) \qquad \rho = \rho_{n,d}\colon (x_0, x_1, y) \mapsto (x_0, e^{\frac{2\pi i}{n}} x_1, y)$$

They have respective order 2 and n, generating a subgroup of $\mathrm{Aut}(C_n)$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

A classical result of Harvey and Wiman ([Har66, Wim95]) is that an automorphisms of a curve of genus g at least 2 has order at most $4g+2$. Moreover, if the equality holds, the curve is $C_{2g+1,1}$, usually refereed in literature as a *Wiman curve.* This is the motivation for the name we chose for these curves.

We recall that

$$H^0(C, K_C) = \left\langle x_0^{\left\lceil \frac{nd}{2} \right\rceil - 2}, x_0^{\left\lceil \frac{nd}{2} \right\rceil - 3} x_1, \ldots, x_1^{\left\lceil \frac{nd}{2} \right\rceil - 2} \right\rangle$$

Of course ι is the hyperelliptic involution acting on $H^0(C, K_C)$ as the multiplication by -1.

Remark 1.2. The "rotation" ρ acts freely out of the locus $x_0x_1 = 0$, corresponding to 3 or 4 points; precisely $4 - [nd]_2$ points.

The divisor $x_1 = 0$ is made by two points, with local coordinate x_1/x_0, where ρ acts locally as multiplication by $e^{\frac{2\pi i}{n}}$.

If nd is even, also the divisor $x_0 = 0$ is made by two points. Notice $\rho(0, 1, y) = (0, e^{\frac{2\pi i}{n}}, y) = (0, 1, e^{-\frac{nd}{2}\frac{2\pi i}{n}} y)$. So, if n is even and d is odd then ρ exchanges the two points: ρ^2 stabilizes them acting on the local coordinate x_0/x_1 as multiplication by $e^{-2\frac{2\pi i}{n}}$. If d is even, both points are stabilized by ρ, acting on the local coordinate x_0/x_1 as multiplication by $e^{-\frac{2\pi i}{n}}$.

Finally, if both n and d are odd, then $x_0 = 0$ is a single point with local coordinate $y/x_1^{\frac{nd+1}{2}}$, so ρ acts locally as multiplication by $e^{\frac{n-1}{2}\frac{2\pi i}{n}}$.

The action of ρ on $H^0(C, K_C)$ can be explicity computed. It will be enough for our purposes to notice that it is of the form

$$x_0^{\left\lceil \frac{nd}{2} \right\rceil -2-a} x_1^a \mapsto e^{(a+\lambda)\frac{2\pi i}{n}} x_0^{\left\lceil \frac{nd}{2} \right\rceil -2-a} x_1^a \tag{1.1}$$

where $\lambda = \lambda(n, d)$ depends only on n and d. In particular

Remark 1.3. The monomials $x_0^{\left\lceil \frac{nd}{2} \right\rceil -2-a} x_1^a$ form a basis of eigenvectors for the action of ρ on $H^0(C, K_C)$.

2. Wiman product-quotient surfaces

Definition 2.1. For all integers n, d_1, d_2 and for all $1 \leq k \leq n-1$ with $\gcd(k, n) = 1$ we define a *Wiman product-quotient surface of type* n, d_1, d_2 with *shift* k to be the minimal resolution S of the singularities of its *quotient model* $X := (C_1 \times C_2)/H$ where

- C_j, $j = 1, 2$ is a generalized Wiman curve of type n, d_j;
- $H \subset \operatorname{Aut}(C_1 \times C_2)$ is the cyclic subgroup of order n generated by the automorphism
$$(x, y) \mapsto \left(\rho_{n,d_1} x, \rho^k_{n,d_2} y\right).$$

Consider the Klein subgroup of $\operatorname{Aut}(C_1 \times C_2)$ generated by the hyperelliptic involutions $(\iota_{n,d_1}, 1)$ and $(1, \iota_{n,d_2})$: the corresponding quotient of $C_1 \times C_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since this group commutes with H and it intersects H trivially, it defines a Klein subgroup $K \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset \operatorname{Aut}(X)$ whose quotient is dominated by $\mathbb{P}^1 \times \mathbb{P}^1$. So X/K is rational.

Lemma 2.2. *The canonical map of S factors through the rational surface X/K.*

Proof. By the Kuenneth formula
$$H^0(C_1 \times C_2, K_{C_1 \times C_2}) \cong H^0(C_1, K_{C_1}) \otimes H^0(C_2, K_{C_2})$$
both involutions $(\iota_{n,d_1}, 1)$ and $(1, \iota_{n,d_2})$ act on $H^0(C_1 \times C_2, K_{C_1 \times C_2})$ as the multiplication by -1.

Since by Freitag Theorem [Fre71, Satz 1] the pull-back map sends $H^0(S, K_S)$ isomorphically onto the invariant subspace $H^0(C_1 \times C_2, K_{C_1 \times C_2})^H$, the holomorphic 2−forms on S cannot separate two points in the same orbit by the action of K. □

In fact, in the “degenerate” case $n = 1$, $S = C_1 \times C_2$ and the canonical map is (if of general type, so if $d_1, d_2 \geq 5$) of the degree 4, the quotient by K. This case is mentioned in [MLP]. This holds also for bigger n.

Theorem 2.3. *Let S be a Wiman product-quotient surface of type n, d_1, d_2, $n \geq 2$.*

(1) If $d_1, d_2 \geq 3$, then K_S is nef.

(2) If $d_1 \geq 4$, $d_2 \geq 5$ then S is of general type with canonical map of degree 4.

The statement is not meant to be sharp. For example, essentially the same proof shows that part (2) extends to the case $d_1 = 3$ with the possible exception $n = 2$.

Proof. To write down explicitly the canonical system of $C_1 \times C_2$, we denote by x_0, x_1, y the coordinates of the weighted projective space containing $C_1 = C_{n,d_1}$ as in Definition 2.1, and by $\bar{x}_0, \bar{x}_1, \bar{y}$ the analogous coordinates for C_2. By the Kuenneth formula the monomials

$$m_{a,b} := x_0^{\lceil \frac{nd_1}{2} \rceil -2-a} \bar{x}_0^{\lceil \frac{nd_2}{2} \rceil -2-b} x_1^a \bar{x}_1^b$$

form a basis of $H^0(C_1 \times C_2, K_{C_1 \times C_2})$ of eigenvectors for the action of the given generator $\left(\rho_{n,d_1}, \rho^k_{n,d_2}\right)$ of H with respective eigenvalues

$$\left(e^{\frac{2\pi i}{n}}\right)^{a+kb+\lambda(n,d_1)+k\lambda(n,d_2)}$$

So a basis of $H^0(S, K_S)$ is given by the monomials

$$\{m_{a,b} | n \text{ divides } a + kb + \lambda(n,d_1) + k\lambda(n,d_2)\} \tag{2.1}$$

(1) We notice, looking at (1.1), that if one monomial $m(x_0,x_1)$ has eigenvalue $e^{t\frac{2\pi i}{n}}$, the "next" monomial $x_1 m(x_0,x_1)/x_0$ has eigenvalue $e^{(t+1)\frac{2\pi i}{n}}$.

In particular, if $d \geq 3$, since $n \leq \lceil \frac{nd}{2} \rceil - 1$, then all n^{th} roots of unity are eigenvalues for the action of $\rho_{n,d}$.

So, if $d_2 \geq 3$, then there is at least one monomial $m_{a,b}$ in the basis of $H^0(S, K_S)$ given in (2.1) for each possible $0 \leq a \leq \lceil \frac{nd_1}{2} \rceil - 2$. Similarly, if $d_1 \geq 3$, there is at least one monomial $m_{a,b}$ for each $0 \leq b \leq \lceil \frac{nd_2}{2} \rceil - 2$.

It follows, since $H^0(C_j, K_{C_j})$ is base point free, that $H^0(C_1 \times C_2, K_{C_1 \times C_2})^H$ has finitely many base points. In particular the fixed components of $|K_S|$ are contained in the exceptional locus of the minimal resolution of the singularities $S \to X$. Then S does not contain any rational curve with selfintersection -1: S is a minimal surface.

(2) By a similar argument if $d \geq 5$ then all n^{th} roots of unity are eigenvalues for the action of $\rho_{n,d}$ with multiplicity at least 2. If $d = 4$, the eigenvalue of the eigenvector $x_0^{\lceil \frac{nd}{2} \rceil -2}$ has multiplicity 2.

So, there are 4 monomials in $H^0(S, K_S)$ of the form $m_{0,b}$, $m_{0,b+n}$, $m_{n,b}$, $m_{n,b+n}$. They map $C_1 \times C_2$ as $x_0^n \bar{x}_0^n, x_0^n \bar{x}_1^n, x_1^n \bar{x}_0^n, x_1^n \bar{x}_1^n$ onto a smooth quadric $Q \subset \mathbb{P}^3$. Then the canonical image of S, dominating Q, is a surface as well.

Choose a general point $q \in Q$. Its preimage in $C_1 \times C_2$ has cardinality $(2n)^2$, giving $4n$ points of S. The Klein group acts transitively on them, giving n points $q_1, \ldots, q_n$ of X/K.

The automorphism $(\rho_{n,d_1}, 1)$ permutes the q_j ciclically and acts on any monomial in (2.1) of the form $m_{1,c}$ as the multiplication by $e^{\frac{2\pi i}{n}}$. So, if q is general enough, $m_{1,c}$ separates the q_j.

□

3. UNBOUNDED SEQUENCES OF WIMAN PRODUCT-QUOTIENT SURFACES

We look at the singularities of X.

By Remark 1.2 the number of points of $C_1 \times C_2$ stabilized by a nontrivial subgroup of h is $(4-[nd_1]_2)(4-[nd_2]_2)$. The group H acts on them with orbits of cardinality 1 or 2, giving a singular locus as follows.

Up to exchange C_1 and C_2 we may restrict to five cases:

(1) d_1 and d_2 are even: in this case X has basket $8\frac{k}{n} + 8\frac{-k}{n}$
(2) n even, d_1 even, d_2 odd: in this case X has basket $4\frac{k}{n} + 4\frac{-k}{n} + 2\frac{k}{n/2} + 2\frac{-k}{n/2}$
(3) n even, d_1 and d_2 odd: in this case X has basket $4\frac{k}{n} + 2\frac{-k}{n/2} + 2\frac{k}{n/2} + 2\frac{-k}{n/2}$
(4) n odd, d_1 even, d_2 odd: in this case X has basket $4\frac{k}{n} + 4\frac{-k}{n} + 2\frac{k(n-1)/2}{n} + 2\frac{-k(n-1)/2}{n}$
(5) n odd, d_1 and d_2 odd: $5\frac{k}{n} + 2\frac{k(n-1)/2}{n} + 2\frac{-2k}{n}$

We only consider case (1), so we assume now that d_1 and d_2 are even.

We notice that the basket is symmetric in the sense that for each singular point of type $\frac{k}{n}$, there is a singular point of type $\frac{-k}{n}$. Then the invariant γ introduced in [BP16, Section 4] vanish and by [BP16, Proposition 4.1]

$$8 - \frac{K_S^2}{\chi(\mathcal{O}_S)} = \frac{l}{\chi(\mathcal{O}_S)} = \frac{l}{\frac{\left(n\frac{d_1}{2}-2\right)\left(n\frac{d_2}{2}-2\right)}{n} + 4\left(1-\frac{1}{n}\right)}$$

where l is the number of irreducible components of the resolution of the singularities $S \to X$.

Writing the continued function of $\frac{n}{k}$ as in (0.1) then ([Rie74, Section 3]) the number of irreducible components of the resolution of two singular points of respective type $\frac{k}{n}$ and $\frac{-k}{n}$ equal $1 + \sum(b_j - 1)$, so

$$8 - \frac{K_S^2}{\chi(\mathcal{O}_S)} = \frac{8\left(1+\sum(b_j-1)\right)}{\frac{\left(n\frac{d_1}{2}-2\right)\left(n\frac{d_2}{2}-2\right)}{n} + 4\left(1-\frac{1}{n}\right)} \approx_{n\to\infty} \frac{32}{d_1 d_2}\frac{1+\sum(b_j-1)}{n}$$

Picking the simplest case $k=1$, then $\frac{1+\sum(b_j-1)}{n} = \frac{1+n-1}{n} = 1$ we deduce

Theorem 3.1. *There is an unbounded sequence S_n of surfaces of general type with canonical map of degree 4 and unbounded $\chi(\mathcal{O}_{S_n})$ such that*

$$\lim_{n\to\infty} \frac{K_{S_n}^2}{\chi(\mathcal{O}_{S_n})} = 8\left(1 - \frac{1}{m}\right)$$

for all positive integer $m \geq 6$ that is not a prime number.

Proof. Write $m = ab$ with $a \geq 2$, $b \geq 3$ and pick the sequence of the Wiman product-quotient surfaces of type $n, 2a, 2b$ and shift 1. □

4. Conclusions

We have constructed countably many unbounded sequences of surfaces with canonical map of degree 4 whose slope tends to different values, by considering only Wiman product-quotient surfaces of type n, d_1, d_2 and shift 1 with d_1 and d_2 even.

A first possible generalization is by changing the shifts. For any sequence of positive integers k_n, with $1 \leq k_n \leq n-1$, $\gcd(k_n, n) = 1$, we can consider the Wiman product-quotient surfaces of type $n, 2a, 2b$ and shift k_n.

Then the sequence S_n of the Wiman product-quotient surfaces of type $n, 2a, 2b$ and shift k_n has

$$\lim_{n\to\infty} \frac{K^2_{S_n}}{\chi(\mathcal{O}_{S_n})} = 8 - 8\frac{1}{m} \lim_{n\to\infty} \sigma\left(\frac{k_n}{n}\right).$$

So answering the question posed in the introduction should allow to construct more sequences of surfaces such that the limit of the slope approach different values.

One can also consider Wiman product quotient surfaces where the d_j are not both even and get similar results. We did part of the computation, showing that the case we did is the best one, in the sense that in all other cases the slope remains higher than $6.\bar{6}$.

One can also extend the definition of generalized Wiman curves by adding hyperelliptic curves of type $y^2 = x_0 x_1 f(x_0^n, x_1^n)$. Our computations shows that this also does not add much. So we decided, for the convenience of the reader, to skip this case, since it would have required a more complicated notation in the description of the group action.

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